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ON A GENERAL RELATION OF CONTINUED FRACTIONS

BY ROBERT E. MORITZ

LET us write with Dirichlet,*

$$(a_1, \dots, a_n) = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}\,; \tag{1}$$

it has then been shown that

$$(a_1, \dots, a_n)(a_n, \dots, a_2) = (a_n, \dots, a_1)(a_1, \dots, a_{n-1}), \tag{2}$$

a relation which Moebius† has characterized as one of the most remarkable relations between continued fractions.

We propose to establish the general relation

$$\prod_{k=k}^{k=l-1}(a_k, \dots, a_n)\prod_{n=n}^{n=m+1}(a_n, \dots, a_l) = \prod_{n=n}^{n=m+1}(a_n, \dots, a_k)\prod_{k=k}^{k=l-1}(a_k, \dots, a_m), \tag{3}$$

$$k < l < m < n,$$

of which (2) is but a very special case. Incidentally we shall arrive at a simple rule for expressing a quotient of two continuants as a product of continued fractions.

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* Dirichlet's *Werke*, vol. 2, p. 141.

† Moebius, *Crelle's Journal*, vol. 6 (1830), p. 218.

Let the continuant

$$p_{r,t} = \begin{vmatrix} a_r & 1 & 0 & . & . & . & . \\ -1 & a_{r+1} & 1 & . & . & . & . \\ 0 & -1 & a_{r+2} & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & a_{t-2} & 1 & 0 \\ . & . & . & . & -1 & a_{t-1} & 1 \\ . & . & . & . & 0 & -1 & a_t \end{vmatrix}, \quad r \underset{<}{=} t, \qquad (4)$$

be developed in terms of the minors of its first $s - r + 1$ rows or columns, and we get

$$p_{r,t} = p_{r,s}\, p_{s+1,t} + p_{r,s-1}\, p_{s+2,t}, \quad r \underset{<}{=} s \underset{<}{=} t. \qquad (5)$$

In order that this equation may not be meaningless when $s = t - 1$, or $s = t$, we must put

$$p_{t+1,t} = 1, \qquad p_{t+2,t} = 0, \qquad (6)$$

which is allowable since $p_{t+1,t}$ and $p_{t+2,t}$ have not yet been defined.

For $s = r$, and $s = t - 1$, we have

$$p_{r,t} = a_r p_{r+1,t} + p_{r+2,t}, \qquad r \underset{<}{=} t, \qquad (7)$$

and

$$p_{r,t} = a_t p_{r,t-1} + p_{r,t-2}, \qquad r \underset{<}{=} t, \qquad (8)$$

respectively.

By properly interchanging the rows and then the columns in (4) and multiplying the elements in each row and column by -1, we have also

$$p_{r,t} = p_{t,r}, \qquad r \underset{<}{=} t. \qquad (9)$$

Equations (7) and (8) furnish immediately a well known relation between continuants and continued fractions; for, dividing (7) by $p_{r+1,t}$, we have

$$\frac{p_{r,t}}{p_{r+1,t}} = a_r + \cfrac{1}{\cfrac{p_{r+1,t}}{p_{r+2,t}}},$$

and in like manner

$$\frac{p_{r+1,t}}{p_{r+2,t}} = a_{r+1} + \cfrac{1}{\cfrac{p_{r+2,t}}{p_{r+3,t}}},$$

. . . . .

. . . . .

$$\frac{p_{t-1,t}}{p_{t,t}} = a_{t-1} + \cfrac{1}{\cfrac{p_{t,t}}{p_{t+1,t}}},$$

$$\frac{p_{t,t}}{p_{t+1,t}} = a_t.$$

Consequently we have by successive substitution

$$\frac{p_{r,t}}{p_{r+1,t}} = (a_r, \cdots, a_t), \qquad r \lesseqgtr t, \tag{10}$$

and similarly we get from (8)

$$\frac{p_{r,t}}{p_{r,t-1}} = (a_t, \cdots, a_r), \qquad r \lesseqgtr t. \tag{11}$$

Since the reciprocal of a continued fraction is again a continued fraction, we may write

$$\frac{1}{(a_r, \cdots, a_t)} = (0, a_r, \cdots, a_t) = (a_r, \cdots, a_t)'. \tag{12}$$

We are now prepared to prove the following proposition:

Every quotient of two continuants, $p_{k,m}/p_{l,n}$, *can be expressed in two different ways as a product of continued fractions of the type* $(a_r, \ldots, a_s)$.

For the sake of simplicity we shall assume for the present that k, l, m, n are all different, furthermore that the quotient $p_{k,m}/p_{l,n}$ is so written that $k < m$, and $l < n$, an assumption which is permissible because of (9). Again, we may assume without loss of generality that $k < l$, for if $k > l$ we need only consider the reciprocal $p_{l,n}/p_{k,m}$ for which then the assumed condition holds. After this reciprocal quotient has been expressed as a product of continued fractions, we get the required expression for the original quotient by changing every primed factor $(a_r, \ldots, a_s)'$ into an unprimed factor $(a_r, \ldots, a_s)$, and *vice versa*.

Because of the foregoing considerations, the twenty-four possible cases reduce to the three following:

(1) $k < l < m < n$,
(2) $k < l < n < m$,
(3) $k < m < l < n$.

Case 1. $k < l < m < n$.

Here we have

$$\frac{p_{k,m}}{p_{l,n}} = \frac{p_{k,m}}{p_{k+1,m}}\,\frac{p_{k+1,m}}{p_{k+2,m}} \quad \cdots \quad \frac{p_{l-1,m}}{p_{l,m}}\,\frac{p_{l,m}}{p_{l,m+1}}\,\frac{p_{l,m+1}}{p_{l,m+2}} \quad \cdots \quad \frac{p_{l,n-1}}{p_{l,n}}.$$

Now equation (10) gives

$$\frac{p_{k,m}}{p_{k+1,m}} = (a_k, \cdots, a_m), \quad \frac{p_{k+1,m}}{p_{k+2,m}} = (a_{k+1}, \cdots, a_m), \text{ etc.},$$

and by (11) and (12)

$$\frac{p_{l,m}}{p_{l,m+1}} = \frac{1}{(a_{m+1}, \cdots, a_l)} = (a_{m+1}, \cdots a_l)', \text{ etc.};$$

hence

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} = {} & (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m) \\ & \times (a_{m+1}, \cdots, a_l)'(a_{m+2}, \cdots, a_l)' \cdots (a_n, \cdots, a_l)', \end{aligned} \tag{13}$$

and similarly

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} & = \frac{p_{k,m}}{p_{k,m+1}}\,\frac{p_{k,m+1}}{p_{k,m+2}} \quad \cdots \quad \frac{p_{k,n-1}}{p_{k,n}}\,\frac{p_{k,n}}{p_{k+1,n}}\,\frac{p_{k+1,n}}{p_{k+2,n}} \quad \cdots \quad \frac{p_{l-1,n}}{p_{l,n}} \\ & = (a_{m+1}, \cdots, a_k)'(a_{m+2}, \cdots, a_k)' \cdots (a_n, \cdots, a_k)' \\ & \quad \times (a_k, \cdots, a_n)\,(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n). \end{aligned} \tag{13'}$$

Case 2. $k < l < n < m$.

In like manner

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} & = (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m) \\ & \quad \times (a_m, \cdots, a_l)(a_{m-1}, \cdots, a_l) \cdots (a_{n+1}, \cdots, a_l) \qquad (14) \\ & = (a_m, \cdots, a_k)(a_{m+1}, \cdots, a_k) \cdots (a_{n+1}, \cdots, a_k) \\ & \quad \times (a_k, \cdots, a_n)(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n). \qquad (14') \end{aligned}$$

An inspection of (13), (13′), (14), (14′) leads to the following *memoria technica* which enables us to write down immediately not only these products, but the products for any quotient of the form $p_{k,m}/p_{l,n}$.

1. *Write down the indices of the elements in the continuant of the numerator in a horizontal line, and under them write the indices of the elements*

in the continuant of the denominator, but in such a manner that each index shall stand under the like index in the first line. Let k and m represent respectively the first and last indices in the upper line, l and n respectively the first and last indices of the lower line. We then obtain the required products by using as factors the following continued fractions:

2. *Those fractions in which the elements, lying between a_k and a_l and including that one of these two which has the lesser index, form the initial elements and a_m (or a_n) the end elements.*

3. *Those fractions in which the elements, lying between a_m and a_n and including that one of these two which has the greater index, form the initial elements and a_l (or a_k) the end elements.*

4. *Prime all those fractions whose initial indices are found in the lower line but not in the upper.*

The equations (13), (13′), (14), (14′) remain true even if some of the indices l, k, m, n are equal. In that case we need only suppress such factors as lose their sense.

For instance, when $l = k$, (14) and (14′) go over into

$$k < n < m,$$

$$\frac{p_{k,m}}{p_{k,n}} = (a_m, \cdots, a_k)(a_{m-1}, \cdots, a_k) \cdots (a_{n+1}, \cdots, a_k), \qquad (15)$$

and for $n = m$, (13) or (13′) gives us

$$k < l < m,$$

$$\frac{p_{k,m}}{p_{l,m}} = (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m). \qquad (16)$$

Of special interest is the case in which the continuant of the denominator reduces to a single element, that is, when in (15) $k = n$, or in (16) $l = m$. Remembering that $p_{k,k} = a_k$ and $p_{m,m} = a_m$, we have

$$p_{k,m} = (a_m, \cdots, a_k)(a_{m-1}, \cdots, a_k) \cdots (a_{k+1}, a_k)\, a_k, \quad k < m, \qquad (17)$$

$$p_{k,m} = (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{m-1}, a_m)\, a_m, \quad k < m, \qquad (18)$$

that is,

Every continuant can be expressed in two ways as a product of continued fractions.

From (17) and (18) it appears immediately that the denominator of any reduced continued fraction on the right equals the numerator of the continued fraction next succeeding it, so that after reduction there remains only the numerator of the first fraction on the right in each equation. We have thus arrived at the well-known theorem:

The continued fractions $(a_m, \cdots, a_k)$ and $(a_k, \cdots, a_m)$ *have equal numerators*;

and to the following simple rule for computing a given continuant:

To find the value of $p_{k,m}$ *compute the numerator of either of the fractions* $(a_k, \cdots, a_m)$ or $(a_m, \cdots, a_k)$.

Case 3. $k < m < l < n$.

Formulae (17) and (18) enable us also to dispose of the remaining third case in the simplest way possible. For we need only write the product for $p_{k,m}$ and $p_{l,n}$ separately and then multiply the fractions of the first by the reciprocals of the factors of the second.

If we equate (13) and (13′) and remember that any primed factor equals the reciprocal of the corresponding unprimed factor we easily deduce

$$\begin{aligned}
&(a_k, \cdots, a_n)(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n) \\
&\qquad \times (a_n, \cdots, a_l)(a_{n-1}, \cdots, a_l) \cdots (a_{m+1}, \cdots, a_l) \\
&= (a_n, \cdots, a_k)(a_{n-1}, \cdots, a_k) \cdots (a_{m+1}, \cdots, a_k) \\
&\qquad \times (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m),
\end{aligned} \tag{19}$$

or, as it may be otherwise written,

$$\prod_{k=k}^{k=l-1} (a_k, \cdots, a_n) \prod_{n=n}^{n=m+1} (a_n, \cdots, a_l) = \prod_{n=n}^{n=m+1} (a_n, \cdots, a_k) \prod_{k=k}^{k=l-1} (a_k, \cdots, a_m). \tag{20}$$

Equations (14) and (14′) furnish essentially the same relation.

Moebius' relation

$$(a_1, \cdots, a_n)(a_n, \cdots, a_2) = (a_n, \cdots, a_1)(a_1, \cdots, a_{n-1})$$

is obtained from (20) by putting $k = 1$, $l = 2$, $m = n - 1$.

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